AIAA Journal

VOLUME 2 AUGUST 1964 NUMBER 8

Particle Motion in the Equatorial Plane

VINCENT L. PISACANE*

Applied Physics Laboratory, The Johns Hopkins University, Silver Spring, Md.

A theoretical formulation has been made to describe the motion of bound particles in the equatorial plane of a concentric gravitational and dipole magnetic field. Exact expressions are obtained for the angular drift, angular drift velocity, and period of the particle about the dipole axis in terms of the normal complete elliptic integrals. More tractable results are obtained by a first-order approximation to the trajectory of the particle. This analysis is applicable to the motion of charged satellites or charged particles in the exosphere. The approximate results of the paper can be extended by using higher-order approximations.

Nomenclature

$\mathbf{a}$ = particle acceleration
a,b,c,d = roots of quartic equation
B = M/r^3, magnitude of dipole magnetic field
I $\equiv \theta_T/2V$
J $\equiv Tv_0/2r_0$
k = gravitational constant
m = mass of particle
n = modulus of elliptic integral
M = magnetic moment of the dipole field
P $\equiv 2k/r_0v_0^2$, ratio of particle potential energy to kinetic energy at r_0
q = quantity of negative charge
$\mathbf{r}$ = position vector of particle from dipole axis
r_0 = radius of the circle whose circumference the trajectory intersects orthogonally
R $\equiv r_0/r$
T = period of the particle over one loop
$\mathbf{v}$ = velocity of the particle
v_c = qM/mr_0^2, velocity of a particle which moves in a dipole field about the dipole axis in a circle of radius r_0
v_g = drift velocity of particle due to uniform gravitational field
v_G = drift velocity of particle due to constant gradient magnetic field
V $\equiv v_c/v_0$
X,Y,Z = inertial reference system
α^2 = $(b-a)/(b-d)$
α_1^2 = $(d/a)\alpha^2$
$\boldsymbol{\varepsilon}_n,\boldsymbol{\varepsilon}_t$ = unit vectors normal to and tangent to velocity vector
θ = angle between position vector and x axis
θ_T = average angular drift over one loop
$\langle d\theta/dt\rangle$ = average angular drift velocity over one loop
φ = angle between position vector and velocity vector
$(\ )_0$ = subscript denoting reference to r_0

Received October 18, 1963; revision received April 28, 1964. This work was supported by the U. S. Department of the Navy, Bureau of Naval Weapons, under Contract NOw 62-0604-c.

* Engineer, Space Research and Analysis Group.

Introduction

THE motion of charged particles in nonuniform magnetic fields has received considerable attention since the discovery of the Van Allen radiation belts. Although the motion is fairly simple to describe qualitatively, it is rather complicated mathematically. The exact integration of the equations of motion is difficult and, in the majority of cases, can be carried out only by numerical means. This, however, is not always practical.

The guiding center approximation for the motion of a charged particle developed by Alfven[1] is very useful when the magnetic field remains sensibly constant spatially over several Larmor radii and temporarily over several Larmor periods. The particle then moves approximately in a circle with a rapidly moving center whose motion can be described in terms of the adiabatic invariants.

Most investigations of the motion of particles trapped in the geomagnetic field utilize the guiding center theory. When the conditions are such that the Larmor radius is not negligibly small as compared with the distance from the dipole axis, errors are introduced by not taking into account the changes in the field and field gradient over the actual path of the particle. The extent of these errors for motion in the equatorial plane of a dipole magnetic field has been treated successfully by Avrett.[2] In this paper, the added complication of a point mass gravitational field is considered, making this analysis applicable to the motion of charged satellites or charged particles in the exosphere. Exact expressions for the angular drift, period, and angular drift velocity in terms of the normal complete elliptic integrals are obtained. A first-order approximation to the trajectory gives more tractable results and a better approximation to the angular drift velocity than that of the guiding center theory.

Equations of Motion

Consider the dynamical system characterized by Fig. 1 in which a negatively charged particle moves under the action

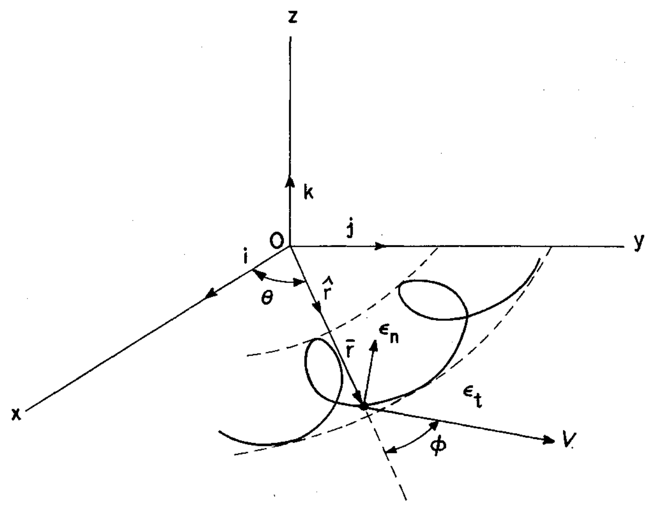

Fig. 1 Geometry of particle motion.

of a magnetic dipole of moment M directed along the negative z axis and a point mass gravitational field located at the origin. The equation of motion of the particle is then

$$-q\mathbf{v} \times \mathbf{B} - mk\mathbf{r}/r^3 = m\mathbf{a} \tag{1}$$

For motion confined to the equatorial plane (x, y),

$$\mathbf{r}/r = -\sin\varphi\boldsymbol{\varepsilon}_n + \cos\varphi\boldsymbol{\varepsilon}_t \tag{2}$$

$$\mathbf{v} = v\boldsymbol{\varepsilon}_t \tag{3}$$

$$d\boldsymbol{\varepsilon}_t/dt = (\dot{\theta} + \dot{\varphi})\boldsymbol{\varepsilon}_n \tag{4}$$

where dots denote ordinary derivatives with respect to time, and the scalar equations derivable from Eq. (1) are

$$qvB/m + k\sin\varphi/r^2 = (\dot{\theta} + \dot{\varphi})v \tag{5}$$

$$-k\cos\varphi/r^2 = \dot{v} \tag{6}$$

Setting

$$B = M/r^3 \tag{7}$$

$$\dot{r} = v\cos\varphi \tag{8}$$

$$r\dot{\theta} = v\sin\varphi \tag{9}$$

Eqs. (5) and (6) become

$$d(\sin\varphi)/dr + \sin\varphi(1/r - k/v^2r^2) = qM/mr^3v \tag{10}$$

$$vdv = -kdr/r^2 \tag{11}$$

Equation (11), when integrated, yields

$$v^2 = v_0^2 + 2k(1/r - 1/r_0) \tag{12}$$

where $v = v_0$ at $r = r_0$. Equation (10) with Eq. (12) reduces to the integrable form

$$df(\varphi)/dr + f(\varphi)\,g(r) = h(r) \tag{13}$$

a first-order linear differential equation with variable coefficients. The solution of Eq. (10) is

$$\sin\varphi = (qM/mr)(1/r_0 - 1/r)(v_0^2 + 2k/r - 2k/r_0)^{-1/2} \tag{14}$$

where, without loss of generality, it is assumed that $\varphi_0 = 0$ at $r = r_0$. Equation (14), an implicit description of the orbit, can be rewritten in the form

$$\sin\varphi = VR(1 - R)/[1 - P(1 - R)]^{1/2} \tag{15}$$

where

$$v_c \equiv qM/mr_0^2 \tag{16}$$

$$V \equiv v_c/v_0 \tag{17}$$

$$R \equiv r_0/r \tag{18}$$

$$P \equiv 2k/r_0v_0^2 \tag{19}$$

The parameter v_c is the velocity with which a particle would move in the dipole field along the circle with radius r_0. The radius of the circle whose circumference is orthogonally intersected by the orbit is r_0. The ratio of the particle's potential energy to kinetic energy at r_0 is P; it provides a measure of the significance of the gravitational field.

Equation (15) can be rewritten in the form

$$R^4 - 2R^3 + R^2 - PR\sin^2\varphi/V^2 + (P - 1)\sin^2\varphi/V^2 = 0 \tag{20}$$

which yields four solutions for the motion of the body. Approximations to the roots of Eq. (20) are

$$R_1 = \tfrac{1}{2}[1 + P\sin\varphi/V(4 - 2P)^{1/2} + \{1 + 2(4 - 2P)^{1/2}\sin\varphi/V - P^2\sin^2\varphi/V^2(4 - 2P)\}^{1/2}] \tag{21}$$

$$R_2 = \tfrac{1}{2}[1 - P\sin\varphi/V(4 - 2P)^{1/2} + \{1 - 2(4 - 2P)^{1/2}\sin\varphi/V - P^2\sin^2\varphi/V^2(4 - 2P)\}^{1/2}] \tag{22}$$

$$R_3 = \tfrac{1}{2}[1 - P\sin\varphi/V(4 - 2P)^{1/2} - \{1 - 2(4 - 2P)^{1/2}\sin\varphi/V - P^2\sin^2\varphi/V^2(4 - 2P)\}^{1/2}] \tag{23}$$

$$R_4 = \tfrac{1}{2}[1 + P\sin\varphi/V(4 - 2P)^{1/2} - \{1 + 2(4 - 2P)^{1/2}\sin\varphi/V - P^2\sin^2\varphi/V^2(4 - 2P)]\}^{1/2}] \tag{24}$$

Equations (21–24) can be rewritten in a more tractable form keeping only first-order values of P:

$$R_1 = \tfrac{1}{2}[1 + P\sin\varphi/2V + \{1 + (4 - P)\sin\varphi/V\}^{1/2}] + O(P^2) \tag{25}$$

$$R_2 = \tfrac{1}{2}[1 - P\sin\varphi/2V + \{1 - (4 - P)\sin\varphi/V\}^{1/2}] + O(P^2) \tag{26}$$

$$R_3 = \tfrac{1}{2}[1 - P\sin\varphi/2V - \{1 - (4 - P)\sin\varphi/V\}^{1/2}] + O(P^2) \tag{27}$$

$$R_4 = \tfrac{1}{2}[1 + P\sin\varphi/2V - \{1 + (4 - P)\sin\varphi/V\}^{1/2}] + O(P^2) \tag{28}$$

For $P = 0$, Eq. (20) admits the exact solutions

$$R_1 = \tfrac{1}{2}[1 + (1 + 4\sin\varphi/V)^{1/2}] \tag{29}$$

$$R_2 = \tfrac{1}{2}[1 + (1 - 4\sin\varphi/V)^{1/2}] \tag{30}$$

$$R_3 = \tfrac{1}{2}[1 - (1 - 4\sin\varphi/V)^{1/2}] \tag{31}$$

$$R_4 = \tfrac{1}{2}[1 - (1 + 4\sin\varphi/V)^{1/2}] \tag{32}$$

to which Eqs. (25–28) also reduce. The solution corresponding to bound motion is given by Eq. (26), where it is required that $V > 4 - 2P$. The equation of the trajectory is then approximated by

$$r = 2r_0[1 - P\sin\varphi/2V + \{1 - (4 - P)\sin\varphi/V\}^{1/2}]^{-1} \tag{33}$$

from which it follows directly that

$$r_{\max} = 2r_0[1 - P/2V + \{1 - (4 - P)/V\}^{1/2}]^{-1} \tag{34}$$

and

$$r_{\min} = 2r_0[1 + P/2V + \{1 + (4 - P)/V\}^{1/2}]^{-1} \tag{35}$$

Angular Drift, Angular Drift Velocity, and Period

The angular drift velocity of the particle about the polar axis is the time average of $d\theta/dt$ given by

$$\langle d\theta/dt\rangle = (\textstyle\int d\theta\cdot dt/dt)/\int dt \equiv \theta_T/T \tag{36}$$

where the integration is over the period during which φ increases by 2π. Utilizing Eqs. (8, 9, 15, and 16), it follows that

$$\theta_T = 2\int_{r_{\min}}^{r_{\max}} V(1-R)[1-P(1-R) - V^2R^2(1-R)^2]^{-1/2}dR \equiv 2VI \quad (37)$$

and

$$T = 2\int_{r_{\min}}^{r_{\max}} (r_0/v_0)R^{-2}[1-P(1-R) - V^2R^2(1-R)^2]^{-1/2}dR \equiv 2(r_0/v_0)J \quad (38)$$

θ_T and T are elliptic integrals expressible in terms of the three standard forms of the elliptic integrals.[3]

The integrands involve the square root of a quartic, and, except in special cases, the roots must be found by numerical means or approximate methods. With the roots known, the radicand is expressible as

$$[1-P(1-R) - V^2R^2(1-R)^2] \equiv [V^2(a-R)(R-b)(R-c)(R-d)] \quad (39)$$

where $a > b > c > d$, all real, and

$$r_{\max} = r_0/b \qquad r_{\min} = r_0/a \quad (40)$$

The method of reducing a general elliptic integral to standard form[3] gives as a result

$$I = C_1K + C_2\Pi(\alpha^2, n) \quad (41)$$

$$J = C_3K + C_4E(\pi/2, n) + C_5\Pi(\alpha_1^2, n) \quad (42)$$

where

$$g = 2[(a-c)(b-d)]^{-1/2} \quad (43)$$

$$\alpha^2 = (b-a)/(b-d) \quad (44)$$

$$\alpha_1^2 = (d/a)\alpha^2 \quad (45)$$

$$n^2 = (a-b)(c-d)/(a-c)(b-d) \quad (46)$$

$$C_1 = g(1-d)/V \quad (47)$$

$$C_2 = -g(a-d)/V \quad (48)$$

$$\gamma = g/Va^2\alpha_1^4 \quad (49)$$

$$C_3 = \gamma[\alpha^4 + (\alpha_1^2 - \alpha^2)^2/2(\alpha_1^2 - 1)] \quad (50)$$

$$C_4 = \gamma[\alpha_1^2(\alpha_1^2 - \alpha^2)^2/2(\alpha_1^2 - 1)(n^2 - \alpha_1^2)] \quad (51)$$

$$C_5 = \gamma[2\alpha^2(\alpha_1^2 - \alpha^2) + (\alpha_1^2 - \alpha^2)^2(2\alpha_1^2n^2 + 2\alpha_1^2 - \alpha_1^4 - 3n^2)/2(\alpha_1^2 - 1)(n^2 - \alpha_1^2)] \quad (52)$$

K, E, Π are the normal complete elliptic integrals of the first, second, and third kinds, respectively, having the forms

$$K = \int_0^{\pi/2} (1 - n^2\sin^2\Omega)^{-1/2}d\Omega \quad (53)$$

$$E = \int_0^{\pi/2} (1 - n^2\sin^2\Omega)^{1/2}d\Omega \quad (54)$$

$$\Pi(\alpha^2, n) = \int_0^{\pi/2} (1 - \alpha^2\sin^2\Omega)^{-1} \times (1 - n^2\sin^2\Omega)^{-1/2}d\Omega \quad (55)$$

Solutions for the angular drift, period, and angular drift velocity can be found to any desired accuracy by numerical means.

Approximate Solutions

For small values of P it is possible to obtain approximate analytical solutions of the elliptic integrals in Eqs. (36–38). The roots of the quartic in the integrals can be approximated by

$$a = \tfrac{1}{2}[1 + P/2V + \{1 + (4-P)/V\}^{1/2}] + O(P^2) \quad (56)$$

$$b = \tfrac{1}{2}[1 - P/2V + \{1 - (4-P)/V\}^{1/2}] + O(P^2) \quad (57)$$

$$c = \tfrac{1}{2}[1 - P/2V - \{1 - (4-P)/V\}^{1/2}] + O(P^2) \quad (58)$$

$$d = \tfrac{1}{2}[1 + P/2V - \{1 + (4-P)/V\}^{1/2}] + O(P^2) \quad (59)$$

Equations (41–55) can then be evaluated more easily from the initial terms of their series expansions in $1/V$ in which only first-order P is retained. These expressions are

$$n^2 = (4/V^2)[1 - P/2 + 8(1-P)/V^2 + 80(1 - 3P/2)/V^4 + \ldots] \quad (60)$$

$$\alpha^2 = -(2/V)[1 + P/2V + 4(1 - 5P/8)/V^2 + 3P/V^3 + 32(1 - 73P/64)/V^4 + 27P/V^5 + \ldots] \quad (61)$$

$$\alpha_1^2 = (2/V^2)[1 - P/2 - 2(1 - 3P/4)/V + 9(1-P)/V^2 - 22(1 - 5P/4)/V^3 + 94(1 - 3P/2)/V^4 + 252(1 - 7P/4)/V^5 + \ldots] \quad (62)$$

$$C_1 = (2/V)[1 + (1 - P/2)/V + (1 - P/2)/V^2 + 4(1 - 7P/8)/V^3 + 7(1-P)/V^4 + 32(1 - 87P/64)/V^5 + \ldots] \quad (63)$$

$$C_2 = -(2/V)[1 + 2(1 - P/4)/V + 8(1 - 3P/4)/V^3 + 64(1 - 5P/4)/V^5 + \ldots] \quad (64)$$

$$C_3 = V[1 + P + P/V - 2/V^2 + P/V^3 - 10(1 - 2P/5)/V^4 + 7P/V^5 + \ldots] \quad (65)$$

$$C_4 = -V[(1+P) - (2+P)/V^2 - 10/V^4 - \ldots] \quad (66)$$

$$C_5 = -P[1 + 2/V + 8/V^3 - 20/V^4 + \ldots] \quad (67)$$

$$I = (3\pi/2V^3)[1 - P/3 + 35(1 - 6P/7)/4V^2 + \ldots] \quad (68)$$

$$J = (\pi/V)[1 + 15(1 - P/3)/2V^2 + 10P/V^3 + \ldots] \quad (69)$$

The initial terms of the expansion for $\langle d\theta/dt\rangle$ are

$$\langle d\theta/dt\rangle = (3v_c/2r_0V^2)[1 - P/3 + 5(1-2P)/4V^2 - 10P/V^3 + \ldots] \quad (70)$$

and other quantities of interest expressed as expansions are

$$(r/r_0)\max = 1 + 1/V + 2(1 - P/4)/V^2 + 5(1 - P/2)/V^3 + 14(1 - 3P/4)/V^4 + \ldots \quad (71)$$

$$(r/r_0)\min = 1 - 1/V + 2(1 - P/4)/V^2 - 5(1 - P/2)/V^3 + 14(1 - 3P/4)/V^4 - \ldots \quad (72)$$

$$(r_{\max} - r_{\min})/r_0 = 2/V + 10(1 - P/2)/V^3 + 84(1-P)/V^5 + \ldots \quad (73)$$

$$(r/r_0)_{\text{av}} = 1 + 2(1 - P/4)/V^2 + 14(1 - 3P/4)/V^4 + \ldots \quad (74)$$

$$\theta_T = (3\pi/V^2)[1 - P/3 + 35(1 - 6P/7)/4V^2 + \ldots] \quad (75)$$

$$T = (2\pi r_0/v_c)[1 + 15(1 - P/3)/2V^2 + 10P/V^3 + \ldots] \quad (76)$$

Comparison of Angular Drift Velocity

To check $\langle d\theta/dt\rangle$ given by Eq. (70), the angular drift velocity of the charged particle is now calculated by superposition of the guiding center orbit theory.[4] The gradient drift velocity is

$$\mathbf{v}_G = -(mv^2\mathbf{B} \times \nabla\mathbf{B}/2qB^3) \quad (77)$$

which yields, for the dipole case,

$$v_G\,|_{r=r_0} = 3v_c/2V^2 \tag{78a}$$

or

$$v_G\,|_{r=r_{av}} = (3v_c/2V^2)[1 + 4(1 - 3P/4)/V^2 + 32(1 - 41P/32)/V^4 + \ldots] \tag{78b}$$

The drift velocity due to the gravitational field is

$$\mathbf{v}_g = -\mathbf{F} \times \mathbf{B}/qB^2 \tag{79}$$

which yields

$$v_g\,|_{r=r_0} = -Pv_c/2V^2 \tag{80a}$$

or

$$v_g\,|_{r=r_{av}} = (-pv_c/2V^2)[1 + 2(1 - P/4)/V^2 + 14(1 - 3P/4)/V^4 + \ldots] \tag{80b}$$

The total guiding center angular drift velocity is then

$$\langle d\theta/dt\rangle\,|_{r=r_0} = (3v_c/2V^2r_0)(1 - P/3) \tag{81a}$$

or

$$\langle d\theta/dt\rangle\,|_{r=r_{av}} = (3v_c/2V^2r_0)[(1 - P/3) + 2(1 - 5P/4)/V^2 + 14(1 - 7P/4)/V^4 + \ldots] \tag{81b}$$

A measure of the error incurred when using the guiding center approximation is then given by

$$\langle d\theta/dt\rangle - \langle d\theta/dt\rangle\,|_{r=r_0} = (3v_c/2r_0V^2)[5(1 - 2P)/4V^2 - 10P/V^3 + \ldots] \tag{82a}$$

$$\langle d\theta/dt\rangle - \langle d\theta/dt\rangle\,|_{r=r_{av}} = (3v_c/2r_0V^2)[-3/4V^2 - 10P/V^3 + \ldots] \tag{82b}$$

Conclusion

An analysis of the bound motion of charged particles in the equatorial plane of a concentric gravitational and dipole magnetic field has been presented. The exact evaluation of the angular drift, angular drift velocity, and period has been reduced to the evaluation of the complete normal elliptic integrals. These results afford a check on the three-dimensional problem when the equations of motion are solved by numerical means.

A first-order approximation to the trajectory valid for small effects of the gravitational field (characterized by small values of P) gives a more tractable form for the angular drift, period, and angular drift velocity about the dipole axis. The first terms of the expansion for the angular drift velocity show that, if the guiding center theory is employed, the field and field gradient should be evaluated at a distance equal to the average distance of the particle from the dipole axis as opposed to r_0. The approximations can be extended to include higher orders of P.

References

[1] Alfven, H., *Cosmical Electrodynamics* (Clarendon Press, Oxford, England, 1950).

[2] Avrett, E. H., "Particle motion in the equatorial plane of a dipole magnetic field," J. Geophys. Res. **67**, 53–58 (1962).

[3] Byrd, P. F. and Friedman, M. D., *Handbook of Elliptic Integrals for Engineers and Physicists* (Springer-Verlag, Berlin, 1954), p. 124.

[4] Spitzer, L., *Physics of Fully Ionized Gases* (Interscience Publishers, Inc., New York, 1956), Chap. 1.